



NBY-003-016401

Seat No. _____

M. Sc. (Mathematics) (Sem. IV) (CBCS) Examination

April / May - 2017

MATHS. CMT - 4001 : Commutative Ring Theory
(Old Course)

Faculty Code : 003

Subject Code : 016401

Time : $2\frac{1}{2}$ Hours]

[Total Marks : 70

Instructions : (1) Answer all the questions.

(2) Each question carries 14 marks.

1 Answer any **seven** :

7×2=14

- (a) Define a *prime* ideal of a ring. Verify that $42\mathbb{Z}$ is not a prime ideal of \mathbb{Z} .
- (b) Define the *Jacobson radical* of a ring R .
- (c) Define *homomorphism of rings*. Illustrate this concept with an example.
- (d) Define *unit* in a ring R . Prove that 1 and -1 are the only units of \mathbb{Z} .
- (e) Define the concept of a *module* over a ring R .
- (f) When is a subset S of a ring R said to be *multiplicatively closed*? Let I be an ideal of a ring R . Show that $\{1+a : a \in I\}$ is a multiplicatively closed subset of R .
- (g) Define a *primary ideal* of a ring R .
- (h) Let $a, b \in \mathbb{Z}$. Prove that $a+ib$ is integral over \mathbb{Z} .
- (i) Define an *Artinian ring*.
- (j) State *going-up* theorem.

2 Answer any **Two** : **2×7=14**

(a) Prove that the nilradical of a ring R is the intersection of all the prime ideals of R .

(b) Let I_1, I_2, \dots, I_n be pairwise comaximal ideals of a ring R . Prove that the ring homomorphism

$$\varphi: R \rightarrow \frac{R}{I_1} \times \frac{R}{I_2} \times \dots \times \frac{R}{I_n} \quad \text{given by the rule}$$

$$\varphi(r) = (r + I_1, r + I_2, \dots, r + I_n) \text{ is surjective.}$$

(c) Let M be a module over a ring R . Prove that M is finitely generated if and only if M is isomorphic to a quotient of R^n for some $n \in \mathbb{N}$.

3 (a) Let p_1, p_2, \dots, p_n be prime ideals of a ring R . Let I be **5**

an ideal of R such that $I \subseteq \bigcup_{i=1}^n p_i$. Prove that $I \subseteq p_i$ for some $i \in \{1, 2, \dots, n\}$.

(b) Let M_1, M_2 be submodules of a module M over a **5**

ring R . Show that $\frac{M_1 + M_2}{M_1} \cong \frac{M_2}{M_1 \cap M_2}$ as R -modules.

(c) Let I be an ideal of a ring R . Prove that $r(r(I)) = r(I)$. **4**

OR

3 (a) Let S be a multiplicatively closed subset of a **5**

ring R . Let $g: R \rightarrow T$ be a ring homomorphism such that

$g(s)$ is a unit in T for all $s \in S$. Prove that there exists

a unique ring homomorphism $h: S^{-1}R \rightarrow T$ such that

$$g(r) = h\left(\frac{r}{1}\right) \text{ for all } r \in R.$$

(b) Let S be a multiplicatively closed subset of a ring R . **5**

Prove that any ideal of $S^{-1}R$ is an extended ideal.

(c) Let F be a field. Let $f: F \rightarrow T$ be a nonzero ring **5**

homomorphism from F into a ring T . Show that f is injective.

4 Answer any **Two** : **2×7=14**

- (a) State and prove the *first uniqueness theorem* on decomposable ideals in a ring R .
- (b) Let $R \subseteq T$ be rings and let T be integral over R . Let q be a prime ideal of T and let $p = q \cap R$. Prove that q is a maximal ideal of T if and only if p is a maximal ideal of R .
- (c) Let M be a module over a ring R . Show that M satisfies ascending chain condition on submodules if and only if every submodule of M is finitely generated.

5 Answer any **Two** : **2×7=14**

- (a) Let R be a nonzero ring. Prove that R has at least one maximal ideal.
- (b) State and prove *Nakayama's lemma*.
- (c) Let I be an irreducible ideal of a Noetherian ring R such that $I \neq R$. Prove that I is primary.
- (d) Let R be an Artinian ring. Show that the nilradical of R is nilpotent.

